ANALYTICAL METHOD FOR SOLVING THE HEAT EQUATION IN FINITE DOMAINS. APPLICATION TO CYLINDERS.

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RESUME. On présente une nouvelle approche pour traiter simplement la conduction multidimensionnelle dans un solide cylindrique. La méthode, basée sur la décomposition de la solution sur les fonctions propres du Laplacien, s'applique pour toutes les conditions limites réalistes que l'on obtient sur la surface d'un cylindre placé dans un écoulement normal à son axe. Elle est donc bien adaptée à l'étude des échanges thermiques dans un échangeur à courants croisés.

1 INTRODUCTION

Let us consider a homogeneous thermal conductive solid cylinder of circular basis, immerse in steady fluid flow, normally to the flow axis, according to figure1**.**Thus, a thermal equilibrium is reached where the thermal boundary condition, at each point M_S, on the cylinder surface (with R_e radius), depends on the angle φ between the cylinder radius OM_e and the asymptotic direction of the flow (see figure1). The length of cylinder axis and the width of the channel where the fluid flows, are very large and there is no heat source in the solid. Then the temperature on the external surface $T_{ex}(\varphi)$ of the solid body is depending on the angle φ only and it is the same for the other thermal surface parameters (convective unit h, components of the heat flux vector **q**). Two interesting geometric configurations maybe retained for the cylinders **:**1) a full solid cylinder, 2) a solid cylinder involving a cylindrical hollow of same axis as the surfaces; in the hollow occurs a fluid flow characterized by negligible gradients in the axis direction (fig1). Usually, the thermal boundary condition on the external wall surface S_e are given under one of the following forms**:** temperature profile on the wall *i.e.* $T(R) = T_{ext}(R,\varphi)$ (a); profile of the heat flux radial component on the wall, *i.e.* $q_r(R) = q_{rext}(R,\varphi)$ (b) ; convective exchange at the wall *i.e.,* $\lambda\ \{\partial T/\partial r\} = h(\varphi)\ \{T(R,\varphi) - T_g(\varphi)\}$ (c). Moreover, in the solid, the thermal conductivity coefficient λ is usually assumed as constant. We can still remark that in the case of the full solid cylinder, the total radial heat flux through any fictitious cylindrical surface involved in the solid material is necessary equal to zero in steady regime; but the local radial heat flux component is not required here to be vanishing because of the two- dimensional effect.

In the case of the cylinder including an hollow, owing to the previous assumptions, the boundary conditions on the external and internal surfaces, respectively S_e (radius R_e) and S_i (radius R_i), may be known also under one of the forms given above. In such conditions the systems under consideration (full or hollow cylinders) present two symmetry planes from both thermal and geometrical points of view**:** namely an horizontal plane, defined by the external flow axis and the cylinder axis, and a vertical plane, containing a cross section of the cylinder and, again, the external flow axis. Fig1 represents the section of the system in a vertical symmetry plane and x'Ox axis is the trace of the horizontal symmetry plane. The configuration described here above is close to those of a heat exchanger involving crossed flows. Thus, it seems interesting to investigate accurately the state of the art about the analytic treatment of such a thermal problem and to develop a method suitable in all the cases described above (possibly coupled with the fluid dynamic studies of the flows). This article is the first step of this research. We develop, in the following sections, a novel approach in order to derive analytic solutions of the heat equation in this domain**:** this new approach will be shown first simpler and more easy to use in the case where other more classical methods have been previously developed. Then it will be shown also that the novel type of analytic calculation may be extended in a larger field than those investigated by using classical approaches.

2- THE STATE OF THE ART.

2.1 Classical approach for the full cylinder

In the classical text-books [1], the case of the full solid cylinder is only explicitly treated, if associated with the external boundary conditions of type (a) above. In [1], one can find, for this case, a method based first on $T_{ext}(R_e,\varphi,z)$ expansion in a Fourier series of φ angle (in our case, the z space coordinate along z axis would not appear). This classical method utilizes then the knowledge of a peculiar solution of this problem, taking a finite value on the cylinder axis **:** namely the function $F_n(r,\varphi,z)$ represented by $I_n(\alpha r)\cos\alpha(\beta-z)\cos n\varphi$, or by the other possible expression $I_n(\alpha r)\cos\alpha(\beta-z)\sin n\varphi$**.** $I_n(\alpha r)$ is the modified Bessel function of *n* order, and the sinusoidal functions are precisely those used to expand the boundary temperature. Thus the solution at each point in the solid is obtained as a linear combination of the infinite countable series of $F_n(r,\varphi,z)$ functions, for varying integer *n,* by adjusting the combination coefficients in order to satisfy the boundary condition. Finally, using algebraic manipulations, the results are transformed in a $J_n(ar)$ first order Bessel functions series. In the case of another boundary condition type, defined above as condition of type b or c, the analytical calculation is not developed, at our knowledge, neither according this method, nor according another else. May be, it would be possible to obtain the result by using the previous approach, but it would be certainly a very hard task, giving the result under a very complicated form Therefore in the next section a novel method will be developed, suitable to give the results using boundary conditions of type a, b or c.

2.2 **Classical approach for the hollow cylinder**

Considering the features of the classical method described above, it is clear that this heavy procedure would be practically impossible to use in the case of an hollow cylinder. As a satisfying coefficient adjustment of the series quoted above would be impossible to obtain **:** because it would be required to verify, in the same time, two different precise conditions, *i.e*., two contradictory constraints. Therefore, usually another classical approach [1], based on conform transformations, is used to treat this case; theoretically the method may be associated to the use of the superposition property of linear system, in order to reduce the real problem to elementary problems involving simpler boundary conditions. But, the unique case where a detailed calculation is made, concerns boundary conditions of type (a) where the surface temperatures are given precisely**:**
$T=0,\ \ for\ r=R_i\ \ on\ \ S_i$ and $T=T_e(R_e,\varphi),\ \ \ for\ r=R_e\ \ on\ S_e$. But the use of the superposition property is here possible only to treat the case where two temperature profiles (different from zero) are given on surfaces S_i and S_e. Otherwise, it would be difficult, probably impossible, to extend this approach to the case where a flux condition (of type b or c) is involved in the boundary conditions. Then, it is impossible to decompose the real physical problem according to partial elementary problems directly solvable by the method**:** the boundary conditions could not represent the values of a same function, respectively equal to zero on S_i and equal to a given profile on S_e. As we show it in next sections, the novel approach proposed in this article furnishes an expansion suitable to be extended whatever the type of boundary conditions previously described .

3. THE THEORETICAL BASES OF OUR APPROACH.

The other authors of classical standard books [2] or of more recent works [3] did not bring significant supplement on this question. The approach presented here has been utilized, to solve equations involving partial derivative functions, in various fields of the physics (notably the diffusion of neutrons). But it was, until to now, poorly employed for solving the heat transfer problems and in this sense, it may be considered as a novel approach in this domain. The aim of this article is to apply and develop this method to the thermal processes described in previous sections.

3.1General features

This approach is mainly based on the spectral properties of the Laplacian operator included in the heat equation: when applied to a space functions defined on a finite spatial domain D the spectre of the Laplacian eigeen functions is a discrete spectrum. If in addition, we require that these functions shall verify, on the frontier of the spatial domain, a "condition of cancellation" (*i.e.,* when on the boundary is verified either the nullity of the function, either the nullity of its normal derivative, or a constant function to derivative ratio) then the countable set of the L eigeen functions forms a basis of the space of the L2 functions defined on the spatial domain D. Moreover each function of the set is orthogonal to the other set functions in the sense of the hermitic product defined on the functional space. Finally the eigeen values associated to the eigeen functions form a discrete countable series of real negative numbers. Then it appears convenient to search the thermal solutions of the heat equation in finite spatial domain, under the form of expansions on the set of the Laplacian eigeen functions. As matter of fact:

--The basis property of these functions guarantees the uniqueness of the solution so found.

--In many cases (especially when the surface and environment temperatures are constant), the boundary condition verified on the eigeen functions insures the thermal boundary conditions, because of their similarity.

-The use of L eigeen function basis allows significant simplifications in the heath equation**:** the Laplacian operator vanishes and the partial derivative equation reduces to a simple differential equation generally easy-to-solve.

But the calculation process described above was rather convenient to treat unsteady phenomena**,** more precisely in geometrical configuration where the multi spatial variable eigeen functions of Laplacian operator are well known. In this theoretical frame the main problem was to describe the coupling between the time dependence and the spatial dependence of the process. .

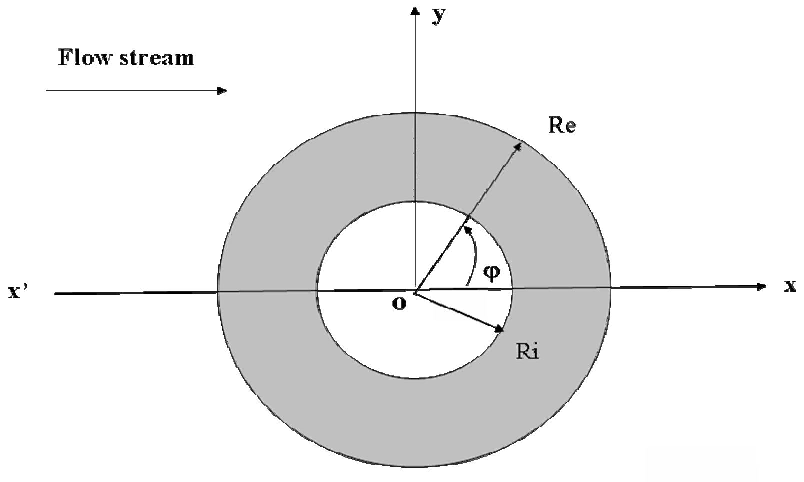
3.2 Special treatment for multi dimensional steady transfers.

In our case the problem is different. In steady condition, without any heat source in the solid mass, the heat equation in the solid reduces to Laplace equation $\Delta T = 0$**,** (where Δ is the Lapacian operator). Thus, here the knowledge of the spectrum of the complete operator is not of any interest. On the contrary it is convenient to split the operator material into different parts regarding respectively the different space variables. It is easily shown, using trivial calculations, that all the properties quoted above regarding the spectral elements of the complete L operator may be transposed on each partial L operator acting on the space of functions restricted to the space variable considered in each partial operator. Therefore the most convenient partial operator will be retained and investigated in regard to its spectral elements. Then the solution will be researched again under the form of a series of functions**:** but here the temperature will be expanded according to the eigeen functions (depending on a single spatial variable) of the retained partial L operator.

4. APPLICATION.TWO-DIMENSIONAL STEADY HEAT CONDUCTION IN FULL AND HOLLOW CYLINDERS. .

The cases of the hollow and full cylinders will be treated together, and the second one will be obtained as the limit of the first one when the radius of the internal **.**solid surface tends to zero. Of course it appears suitable to use here the cylindrical coordinates: r (the local radius) and φ (thepolar angle).Consequently equation the Laplace equation reads in a more explicit form**:**

$$r\{\partial / \partial r(r\partial T / \partial r)\} + \partial^2 T / \partial \varphi^2 = 0 \qquad (1)$$



Moreover, whatever the boundary conditions of type a, b or c considered in the previous sections, taking into account the horizontal plane of symmetry of the problem (defined by x'Ox in figure1), it is clear that the boundary conditions on the φ angle will be:

Fig.1 The cylinder axes are normal to the plane of the sheet

$$\varphi = 0 \;\rightarrow\; q_{\varphi}(r,0) = -(\lambda / r)\{\partial T / \partial\varphi\}_0 = 0\,;\; \varphi = \pi \rightarrow q_{\varphi}(r,\pi) = -(\lambda / r)\{\partial T / \partial\varphi\}_{\pi} = 0 \quad (2)$$

which represent a type of "cancellation condition" described in section 3.1.
Let us note first, that from now, we reduce the domain of the study to the superior half-cylinder (corresponding to $\varphi \in [0,\ \pi]$) and we complete then the study by using the symmetry quoted previously. Let us note then, that here the physical boundary conditions on the r space variable are not convenient here to play the role of "cancellation boundary conditions. Therefore, according to the comments at the end of sections 3-1 and 3-2 on the suitability of the of the basis, it appears here more convenient to chose as basis functions a set of function depending on the φ variable, namely the eigeen functions of the partial L operator depending on φ in Eq. (1). Therefore we will write the temperature in the following form:

$$T = \sum_{0}^{+\infty} C_n(r) f_n(\varphi) \quad , \quad d^2 f_n / d\varphi^2 = \lambda_n f_n \quad (df_n / d\varphi)_0 = (df_n / d\varphi)_{\pi} = 0 \quad (3)$$

where the $f_n(\varphi)$ are the set of eigeen of functions of the partial L operator $\partial^2 / \partial\varphi^2$ present in Eq. (1) and where the latter condition reproduces the conditions (2). Then, solving the system (3) and normalizing the solutions on the $[0-\pi]$ range described by φ when the upper half cylinder is described, one easily obtains:

$$f_0 = 1/\sqrt{\pi} \qquad f_n = \sqrt{2/\pi}\cos n\varphi \quad , \quad \lambda_n = -n^2 \quad (4)$$

where n describes the natural integer values. Then, injecting expression (3) of the temperature in Eq.(1) and taking into account the basis properties we obtain , for $n \neq 0$, a second order homogeneous differential equation:

$$d / dr\{r d / dr\} = 0 \;\; for \;\; n = 0 \;\text{(a)} \;;\; r^2(d^2 C_n / dr^2) + r(dC_n / dr) - n^2 C_n = 0 \;\text{(b)} \quad (5)$$

Eq. (5a) leads obviously to $C_0 = A_o + B_0 Lnr$ As well known, for $n \neq 0$, a typical solution of Eq.(5b) is searched under the form: $C_n = r^s$, where s is a real number to be determined . Replacing the C_n expression in (5b) yields:

$$s^2 - n^2 \;\rightarrow\; s = \pm n \;\;\text{--->}\;\; C_n = A_n r^n + B_n r^{-n} \;\; , \quad (6)$$

where A_n and B_n are sets of real constants depending on the thermal boundary conditions.
Using non dimensional spatial variables: $\rho = r/R_e$, $\rho_i = r_i/R_e$, the general solutions of the family of problems described above, reads:

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$$\theta = T/T_{ref} = \sqrt{1/\pi}\ \{a_0^* + b_0^* Ln\rho\} + \sqrt{2/\pi}\ \sum_n \{a_n^*\rho^n + b_n^*\rho^{-n}\}\cos n\varphi \qquad (7)$$

with $\rho \in [\rho_i\ 1]$, where the choice of T_{ref} depends on the thermal conditions of the problem.

In the same way, the calculation of the constants a_n^* and b_n^* (like would be those of A_n and B_n) is carried out from the boundary conditions on the cylindrical surfaces limiting the solid.
From now, ***we consider especially the case where the temperature profiles are prescribed on the cylindrical boundary surfaces.*** Using the non dimensional space variables we note:

$$T(\rho_i,\varphi) = T_i\ \theta_i(\varphi)\ \ on\ S_i \qquad\qquad T(1,\varphi) = T_e\ \theta_e(\varphi)\ on\ S_e \qquad (8)$$

To manage more easily the calculation we use the steady state superposition property characterizing the problem. The real physic problem is split into two partial problems

$$T(\rho,\varphi) = u(\rho,\varphi) + v(\rho,\varphi), \qquad (9)$$

where u and v have also to verify simultaneously:

$$u(\rho_i,\varphi) = 0 \quad v(\rho_i,\varphi) = T_i.\ \theta_i(\varphi)\ \ ;\ \ u(1,\varphi) = T_e.\theta_e(\varphi) \quad v(1,\varphi) = 0\,. \qquad (10)$$

Moreover, at any point in the solid, according to Eqs.(1, 3a), we obtain for u:

$$r\{\partial/\partial r(r\partial u/\partial r)\} + \partial^2 u/\partial\varphi^2 = 0 \qquad\qquad u = \sum_0^{+\infty} C_{u,n}(r) f_n(\varphi)\,, \qquad (11)$$

Then considering the uniqueness of the solution it is clear that in any point T will be obtained according to Eq.(9). Indeed, from Eq (11) we obtain also according to Eq.(6):

$$C_{u,0} = A_{u,0} + B_{u,0} Lnr = a_{u,0} + b_{u,0} Ln\rho \qquad C_{u,n} = A_{u,n} r^n + B_{u,n} r^{-n} = a_{u,n}\rho^n + b_{u,n}\rho^{-n} \qquad (12)$$

Then, from Eqs.(10), using the property of the basis and the hermitic product defined on the L2 functional space, we obtain, respectively on S_i and S_e, for *u*:

$$a_{u,0} = -b_{u,0} Ln\rho_i\,; \qquad a_{u,n}\rho_i^n = -b_{u,n}\rho_i^{-n} = k_n, \quad if\ \ n \neq 0 \qquad (13a)$$

$$a_{u,0} = T_e/\sqrt{\pi}\ \int_0^{\pi}\theta_e(\varphi)d\varphi, \qquad a_{u,n} + b_{u,n} = T_e\sqrt{2/\pi}\ \int_0^{\pi}\theta_e(\varphi)\cos n\varphi\ d\varphi,\ \ if\ n \neq 0 \qquad (13b)$$

Solving Eqs..(13) using Eqs..(11,12) , we obtain *u*

$$u = -(T_e)\ ([\overline{\theta_{e,0}}\{Ln(\rho/\rho_i)\}/Ln\rho_i] + 2\sum_{n=1}^{+\infty}\overline{\theta_{e,n}}\{(\rho/\rho_i)^n - (\rho/\rho_i)^{-n}\}\cos n\varphi/\{\rho_i^n - \rho_i^{-n}\}) \qquad (14)$$

Where the following average temperatures have been defined whatever *n* **:**
$\overline{\theta_{e,n}} = (1/\pi)\int_0^{\pi} \theta_e(\varphi)\cos n\varphi \, d\varphi \ ,\forall n$ **.** Of course, using the same way for v, we obtain **:**

$$v = (Ti) \ ([\overline{\theta_{i,0}}\{Ln(\rho)\}/Ln\rho_i] + 2 \sum_{n=1}^{+\infty} \overline{\theta_{i,n}}\{(\rho)^n - (\rho)^{-n}\}\cos n\varphi / \{\rho_i^n - \rho_i^{-n}\}) \qquad (15)$$

where $\overline{\theta_{i,n}}$, is deduced from $\theta_i(\varphi)$ in the same way as $\overline{\theta_{e,n}}$ is deduced from $\theta_e(\varphi)$. Then we use Eq.(9), we note $\theta^* = T/(T_i + T_e)$, we remark that the difference $\{(\rho)^n - (\rho)^{-n}\}$ reads also $2Sh(nLn\rho)$; thus writing the other similar expressions in the same way, we can write:

$$\theta^* - \theta_0^* = 2\sum_{n=1}^{+\infty}[\frac{\overline{\theta_{i,n}^*}Sh(nLn\rho) - \overline{\theta_{e,n}^*}Sh(nLn(\rho/\rho_i))}{Shn(Ln\rho_i)}]\cos n\varphi = 2\sum_{n=1}^{+\infty}\theta_n^* \qquad (16)$$

$\overline{\theta_{e/i,n}^*}$ is deduced from $\theta_{e/i,n}^*$ in the same way as $\overline{\theta_{e/i,n}}$ is deduced from $\theta_{e/i,n}$ and θ_0^* the first term, of order *n=o,* in the θ^* expansion, equals $\{\overline{\theta_{i,0}^*}Ln(\rho) - \overline{\theta_{e,0}^*}Ln(\rho/\rho_i\}/Ln\rho_i$. This form is not the same as those found in reference [CJ], but such a result is not surprising because the sets of functions generating the functional space are numerous and each of them leads to a different expansion of the solution and so to a different form. We can still note the intriguing following result: each term θ_n^* of the sum Σ in (25) second member tends to θ_0^* when n tens to 0.
In addition, when $\rho_i \rightarrow 0$, *i. e.* for the full cylinder case, each term θ_n^* of the sum tends to $\overline{\theta_{e,n}^*}\ \rho^n \cos n\varphi$ (which could be obtained directly solving (10b) in this case) and moreover Eq. (10a) would lead to θ_0^* equal to $\overline{\theta_{e,n}^*}$ **.** But here it is convenient to use anywhere the non dimensional temperature $\theta = T/T_e$ instead of θ^*, so we obtain

$$\theta = \overline{\theta_{e,0}} + 2\ \overline{\theta_{e,n}}\ \rho^n \cos n\varphi \qquad (17)$$

COMMENTS AND CONCLUSION.

This method may be used for the other forms of boundary conditions quoted in the Introduction. For example when a normal heat flux is given on a surface the eigeen functions of L operator remain the same. Moreover the boundary conditions on the surfaces are treated in a similar manner as previously: writing the boundary conditions on the surface by mean of equations analogous to Eq.(10). The temperature expressions have just to be previously derived following the *r* (or ρ) variable. Finally considering realistic boundary conditions of type (c), no supplementary theoretical difficulty appears. Thus to solve a crossed flow heat exchanger problem we have just to obtain realistic simple expression of h_{conv} the convection coefficient per surface unit. This is our aim in next future.

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