OPTIMAL ALLOCATION OF HEAT TRANSFER INVENTORY IN ENDOREVERSIBLES CYCLES AND CASCADES

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ABSTRACT: Wherever effective heat transfer area and global heat transfer coefficients are not known precisely, and they seldom are, heat exchanger inventory can be analyzed in terms of heat conductance instead of heat transfer area. Power optimization in endoreversible cycles and cascades consists in optimally allocating a fixed temperature potential between power production and heat transfer processes for a given heat conductance inventory. Solutions of these problems have been shown to allocate the available heat conductance equally among the heat exchangers with a Newtonian heat transfer law. This remarkable property is here generalized by means of an electrical analogy for endoreversible cycles and power cascades. It is shown that this method of solution establishes the equal distribution of heat conductance in all heat exchangers as a general rule to greatly simplify the analysis and the solution of endoreversible cycles and cascades of such cycles.

RESUME : Lorsque les surfaces et les coefficients d'échange ne sont connus séparément que de façon approximative, comme ils le sont pratiquement toujours, les inventaires de surfaces d'échange peuvent être analysés en termes de conductance thermique. L'optimisation de la puissance des cycles endoréversibles et cascades de tels cycles consiste à allouer de façon optimale le potentiel de température donné entre la production de puissance et les transferts thermiques pour une conductance thermique spécifiée. Les solutions de ces problèmes montrent qu'à l'optimum, la conductance thermique totale est répartie également entre les échangeurs pour un modèle de transfert newtonien. Cette propriété remarquable est ici généralisée au moyen d'une analogie électrique pour les cycles endoreversibles et les cascade de puissance. On montre que cette méthode de solution généralise l'équipartition de la conductance globale entre les échangeurs simplifiant ainsi de façon remarquable l'analyse et la solution des problèmes d'optimisation des cycles endoréversibles et les cascades de puissance de tels cycles.

KEYWORDS: F.T.T., maximum power cascade, endoreversible cycles, heat conductance allocation.

NOMENCLATURE: SI units

A	heat exchanger area	(m^2)
C_c	condenser thermal conductance $\equiv (UA)_c$	(W/K)
C_h	boiler thermal conductance $\equiv (UA)_h$	(W/K)
C_T	cycle overall thermal conductance $\equiv (UA)_T$,	(W/K)
$\dot{P}$	mechanical power	(W)
$\dot{Q}_h$	boiler heat rate	(W)
$\dot{Q}_c$	condenser heat rate	(W)
$\dot{Q}_I$	intermediate heat exchanger heat rate	(K)
T_h	hot source isotherm	(K)
T_{ce}	simple cycle condenser isotherm	(K)
T_{ce1}	top cycle lower isotherm	(K)
T_{ce2}	bottom cycle upper isotherm	(K
T_h	hot source isotherm,	(K)
T_{hc}	simple cycle boiler isotherm	(K)
T_{hc1}	top cycle boiler isotherm	(K)
T_{hc2}	bottom cycle boiler isotherm	(K)
U	heat transfer coefficient	($W/m^2.K$)
η	cycle or cascade efficiency	
η_{C-A}	maximum power efficiency	

1. INTRODUCTION

Several optimal solutions have been reported en recent years regarding endoreversible power and refrigeration cycles, including those of Bejan [1, 2, 3], which consider an overall heat transfer conductance to be allocated between a heat source and a sink. Some elaborate algebra had to be used by the authors to obtain an analytical solution to the extremum of the non constrained objective function because of the non linear, coupled necessary condition equations. The optimal heat conductance allocation arrived at is an equipartition, both in the maximum power cycle as well as in the refrigeration cycles. The maximum power solution is then worked out as proportional to one fourth of the overall heat transfer conductance.
Similarly, in his thermoelectric generator analogy, Gordon [4] expresses the maximum obtainable power as proportional to one fourth of the total conductance allocated to the two dissimilar material branches connected in parallel between the reservoirs temperatures T_h and T_c.
Considering two stage combined refrigeration cycles, endoreversible and irreversible, with constant temperature differences and a total heat transfer area some authors [5, 6] gave results which are consistent, although not explicitly expressed as such, with the equal distribution of the given heat transfer conductance among the three heat exchangers, i.e. one third for each exchanger when a unique overall heat transfer coefficient is adopted.
Power cascades are a useful concept to consider for the purpose of regasefying LNG while producing mechanical power [7]. To maximize the cascade potential power amounts to distributing optimally the total available temperature potential and the heat transfer inventory between heat transfer and power production processes.
These results suggest that some generalized rule for optimal heat conductance allocation could be obtained by means of an electrical analogy adapted from the maximum power transfer theorem of electronics [8]. This rule says that in an electrical circuit comprising a signal generator and a signal receiver, the signal intensity received achieves a maximum when the impedances of the generator and receiver are equal. That is, in such a circuit, the signal reaches a maximum value when the two impedances fed in parallel are equal; in this instance the total conductance is the sum of the two parts. Under similar conditions, an endoreversible cycle defined with a constrained thermal conductance to be allocated between the heat source and the sink can be considered as the thermal analog of this electric circuit in which conductances are arranged in series.

2. PROBLEM DEFINITION

A simple electrical analog is represented in figure 2 for the endoreversible power cycle of figure 1 where heat source and sink have infinite thermal capacities; constant temperature differences are necessary assumptions for the electrical analogy with single value voltages. The maximum power problem definition for an endoreversible cycle is then formulated as follows.

Max $\dot{P} = \dot{Q}_h - \dot{Q}_c$ (1)

Subject to the following constraints:

$\dot{Q}_h/T_{hc} - \dot{Q}_c/T_{ce}=0$ (2)

$\dot{Q}_h/(T_h-T_{hc}) + \dot{Q}_c/(T_{ce}-T_c)=C_T$ (3)

This optimization problem is seen to be defined in terms of four unknowns: the two heat rates, $\dot{Q}_h$ - $\dot{\mathbf{Q}}_\mathbf{c}$ and the two fluid isotherms T_{hc} and T_{ce} .While the cycle is endoreversible, the entropy flux is not constant since entropy is generated during heat transfer.
The proposition is made here that in an endoreversible cycle connected to the external heat source and sink temperatures, respectively T_h and T_c, power is maximized when the total conductance C_T

composed of the individual conductances in series is maximized; this situation corresponds to the overall equivalent thermal resistance being minimized.
However, the total available temperature potential (T_h –T_c) between the heat source and the heat sink remains to be allocated between power production and heat transfer processes; the temperature power potential (T_{hc}-T_{ce}) is considered to be the analog of a counter electromotive force (c.e.m.f) achieved by an electrical generator. Because these processes of heat transfer and power production are seen to be uncoupled, a solution by an electrical analogy is proposed, where the optimal allocation heat conductance is solved first, then the maximum power production is addressed next in a much simplified and straight forward approach. Previous work shows that solving this maximum power problem with Lagrangian methods becomes cumbersome often eluding explicit solutions, compared to the simplicity provided by the electric analog method proposed here.

3. ELECTRICAL ANALOGY

The electrical analogy considered in figure 2 is for an endoreversible power cycle for which the source temperature is T_h and the sink temperature T_c. The source supplies $\dot{Q}_h=C_h(T_h-T_{hc})$ and the heat sink receives $\dot{Q}_c=C_c(T_{ce}-T_c)$. The analogy is based on the conservation of electrical current. The endoreversible power cycle between temperatures T_{hc} and T_{ce} is modeled with a current

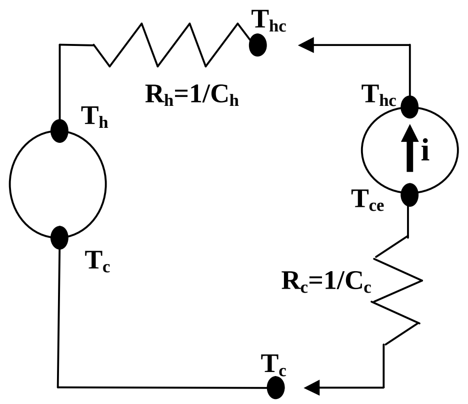

Figure 2: Thermal-electrical analog of the maximum electric power transfer theorem

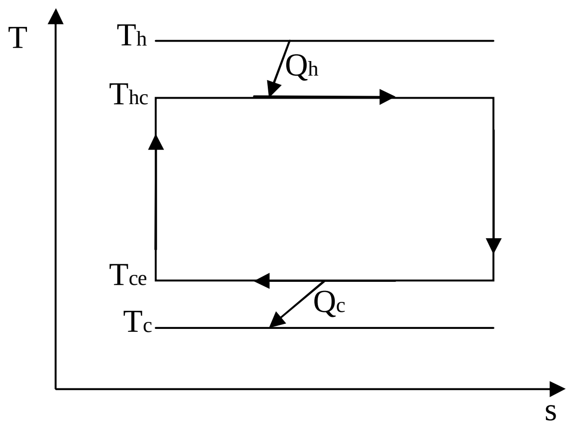

Figure 1: T-s diagram of an endoreversible power cycle

generator which delivers $\dot{Q}_h$ - $\dot{Q}_c$ counter clockwise; the net current in the circuit is clockwise and equal to $\dot{Q}_c$. The electrical current flows from port T_h to port T_c, through the resistors in series, R_h in the left circuit and R_c in the right circuit. Increasing R_c would decrease the current and increase the voltage across R_c, thus decreasing the power dissipated by Joule effect. Conversely, decreasing R_c would increase the current intensity and decrease the voltage across R_c. A certain value R_c with respect to R_h will result in a maximum power dissipation in R_c. In such a circuit, the maximum power dissipated in R_c occurs when R_h equals R_c. This situation is desired in electronics when it is necessary to transfer maximum power through matched internal and external impedances.
Given a voltage source analog to ΔT, the power delivered by this source to the circuit is equal to the voltage ΔT squared divided by the total resistance of the circuit. For the case where two resistances are in series, R_h and R_c, the sum of which is R, the maximum power problem is defined as:

$$\text{Max } \dot{P}= (\Delta T)^2/R=(\Delta T)^2[1/C_c+1/(C_T-C_c)]^{-1} \qquad (4)$$

The total equivalent resistance R is equal to the sum of the inverse conductances; maximizing the power amounts to minimizing the sum of the inverse conductances of equation (4) with respect to C_c. Applying the necessary and sufficient conditions for the existence of a minimum, it can be shown that maximum power is achieved when the total available conductance C_T is distributed equally among C_c and C_h.

$$C_c=C_h=C_T/2 \tag{5}$$

This result is easily extended to any number of resistances in series in an electrical circuit. In the case of a two endoirréversible cycle power cascade, the diagrams of which are given in figures 3 and 4, there are three conductances in series the sum of which is C_T; the optimal conductance allocation is achieved when each conductance is equal to one third of the given conductance C_T. The maximum power actually to be achieved will depend on the applicable temperature potential ΔT. Power maximization with respect to fluid cycle isotherms is addressed next using the equal partition of heat conductance rule obtained here by analogy.

4. MAXIMUM POWER OF AN ENDOREVERSIBLE CYCLE

Revisiting the known solution of this problem [1] illustrates the method of solution presented here. Using the optimal heat transfer conductance allocation obtained above, the maximization problem consists in maximizing $\dot{P} = \dot{Q}_h - \dot{Q}_c$. But the power can now be simply expressed as:

$$\dot{P} = (C_T/2)[(T_h-T_{hc}) -(T_{ce} -T_c)]= (C_T/2)[(T_h+T_c)-(T_{hc} +T_{ce})] \tag{7}$$

Maximizing $\dot{P}$ amounts to minimizing the sum ($T_{hc}+T_{ce}$) subject to the reversibility constraint. This minimization problem is then defined by:

$$\text{Min } \varphi \equiv T_{hc} + T_{ce}$$
$$\text{s.t. } T_h/T_{hc} + T_c/T_{ce}=2 \tag{8}$$

After substitution into the objective function for T_{ce} and applying the necessary condition for an extremum with respect to T_{hc}, one obtains the optimum fluid cycle top isotherm:

$$T_{hc}=T_c\, R(R+1)/2 \tag{9}$$

Where R is defined by the heat source and sink square root temperature ratio:

$$R\equiv(T_h/T_c)^{1/2} \tag{10}$$

Replacing in the constraint, one obtains the optimum fluid cycle bottom isotherm:

$$T_{ce}=T_c\,(R+1)/2 \tag{11}$$

From these one obtains the maximum power and the corresponding efficiency:

$$P_{max}=(C_T/4)\, T_c(R-1)^2 \tag{12}$$

$$\eta_{Pmax}=1 -1/R \tag{13}$$

We now turn to the solution of a power cascade made of two endoreversible cycles.

5. MAXIMUM POWER OF A TWO ENDOREVERSIBLE CYCLE CASCADE

Figure 3 is an electrical analog of the power cascade made of two endoreversible cycles of figure 4. Here we seek to maximize the power P of a two endoreversible cycle cascade with respect to the cascade isotherms T_{hc1}, T_{ce1}, T_{hc2}, T_{ce2}. Using the equal heat conductance allocation among the three heat exchangers, $C_h=C_i=C_c=C_T/3$, the cascade power is expressed by:

$$\dot{P} = (C_T/3)\,[(T_h-T_{hc1}) - (T_{ce2}-T_c) = (C_T/3)[(T_h+T_c) - (T_{hc1}+T_{ce2}) \qquad (14)$$

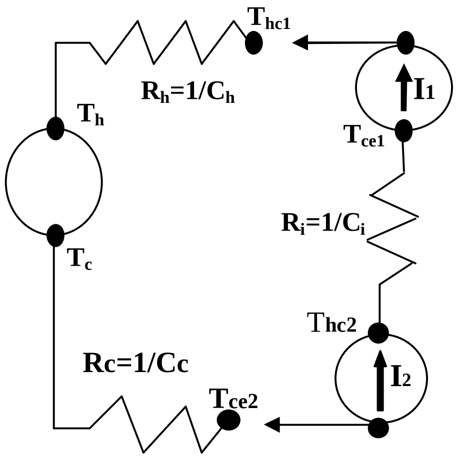

Figure 3: Electrical analog of a two endoreversible cycle power cascade

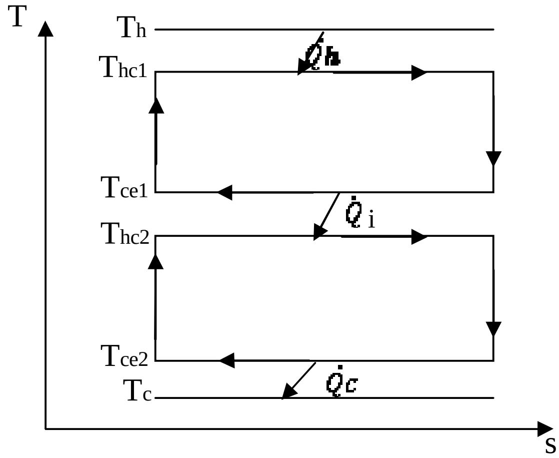

Figure 4: T-S diagram of a two endoreversible cycle power cascade

Maximizing P amounts to minimizing the sum of the cascade external isotherms ($T_{hc1}+ T_{ce2}$) subject to the two constraints of endoreversibility. This minimization problem is then defined by:

$$\begin{aligned}&\text{Min } \varphi \equiv T_{hc1}+T_{ce2}\\&\text{Such that:} T_h/T_{hc1} + T_{hc1}/T_{ce1}=2\\&T_{ce1}/T_{hc2} + T_c/T_{ce2}=2\end{aligned} \qquad (15)$$

In this problem, there are four variables, T_{hc1}, T_{ce1}, T_{hc2}, T_{ce2} and two constraints, hence two degrees of freedom. We solve the two constraints for T_{hc1} and T_{ce2} and replace them in the objective function; we obtain a non constrained objective in terms of the ratio T_{ce1}/T_{hc2}.
Applying the necessary conditions with respect to T_{ce1} and T_{hc2} yields the same equation in terms of the ratio T_{ce1}/T_{hc2} which is obtained as:

$$T_{ce1}/T_{hc2}= (2R+1)/(R+2) \qquad (16)$$

Using the two constraints (15) and this result, one obtains the cascade external isotherms T_{hc1} and T_{ce2}, the maximum power and the corresponding efficiency of the cascade.

$$T_{hc1}=(T_c/3)R(2R+1) \qquad (17\text{-}1)$$
$$T_{ce2}=(T_c/3)(R+2) \qquad (17\text{-}2)$$

The optimum cascade heat rates, maximum power and corresponding efficiency are, respectively:

$$\dot{Q}_h= (C_T/3)[(T_h-T_{hc1})= R(R-1).(C_T.T_c)/9 \qquad (18)$$
$$\dot{Q}_c= (C_T/3)[(T_{ce2}-T_c)= (R-1).(C_T.T_c)/9 \qquad (19)$$

$$\dot{P}_{max} = \dot{Q}_h - \dot{Q}_c = (R-1)^2.(C_T.T_c)/9 \quad (20)$$
$$\eta_{Pmax} \equiv \dot{P}_{max} / \dot{Q}_h = 1 - 1/R \quad (21)$$

These results are remarkable in the sense that the maximum power cascade efficiency is the same as that of a simple endoreversible cycle at maximum power. It is determined by solely by equation (21). This can be considered as a generalization of the Curzon-Ahlborn [9] maximum power efficiency. However, maximum power of a two cycle cascade is reduced to four ninths of that of the simple cycle. The following optimality rules give the cascade efficiency and maximum power for any number of endoreversible cycles by simple inspection:
A. At maximum power, the efficiency of a cascade of any number of endoreversible cycles with heat source and sink temperatures T_h and $T_{c;}$ is that of Curzon-Ahlborn, that is: **η_{Pmax} =1 -1/R**
B. The maximum power of a cascade of N endoreversible cycles: Pmax=$C_T.T_C(R-1)^2/(N+1)^2$. It is reduced by the factor 4//$(N+1)^2$ with respect to that of a single endoreversible cycle.

6. CONCLUSION

The electrical analogy of maximum power theorem is used in this work to simplify the analysis and mathematical solution of the maximum power of an endoreversible cycle or cascades of such cycles with a fixed heat transfer inventory. While known solutions of some of these problems have been arrived at with tedious algebra and lake of generalization, the method presented here uncouples the problem dependency on temperature and heat conductance and allows a simple unconstrained maximization solution. The proposed generalization rules say that maximum efficient of a cascade of any number of endoreversible cycles is the same as that of Curzon-Ahlborn efficiency of an endoreversible cycle, but its power per unit conductance is reduced by a known decreasing factor of the number of cycles. From these results, power cascades efficiencies for LNG regaseification are expected to be in the 25-30% range, slightly below simple gas turbine cycle efficiencies.

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